

CHERN-SIMONS FORMS FOR \mathbb{R} -LINEAR CONNECTIONS ON LIE ALGEBROIDS

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ABSTRACT. The Chern-Simons forms for \mathbb{R} -linear connections on Lie algebroids are considered. A generalized Chern-Simons formula for such \mathbb{R} -linear connections is obtained. We it apply to define Chern character and secondary characteristic classes for \mathbb{R} -linear connections of Lie algebroids.

1. INTRODUCTION

We observe that non-linear objects (forms, connections, mappings between modules of cross-sections of vector bundles, which are non-linear over a ring of smooth functions) have increasing meaning in problems of differential geometry. S. Evens, J. H. Lu and A. Weinstein considered especial non-linear connections of Lie algebroids called connections up to homotopy (see [7]). Crainic and Fernandes [5], [6] introduce the Chern character for non-linear connections. They discuss non-linear forms on Lie algebroids with values in a super vector bundle as antisymmetric, multilinear maps over \mathbb{R} (not necessarily multilinear over the ring of smooth functions), which have a local property. Every non-linear connection ∇ establishes on non-linear forms the covariant derivative operator. If ∇ is flat, the Chern character vanishes and the induced covariant derivative operator is the exterior derivative, and in classically way defines the cohomology space. Crainic and Fernandes introduced secondary characteristic classes for connections up to homotopy [5], [6]. We stay the question whenever these ideas refer to \mathbb{R} -linear forms and \mathbb{R} -linear connections – meaning as objects for which it is not supposed a local property. In the paper, using the generalized Stokes formula for \mathbb{R} -linear connections on Lie algebroids, we prove the Chern-Simons transgression formula without assumption locality for \mathbb{R} -linear connections. This is a helpful starting point to define characteristic classes for \mathbb{R} -linear connections on Lie algebroids. Some Crainic and Fernandes ideas we use to extend notions of Chern character and exotic (secondary) characteristic classes to \mathbb{R} -linear objects. Moreover, we found some explicit formulae for \mathbb{R} -linear Chern-Simons forms. In particular, we gain an direct formula of exotic (secondary) characteristic classes for an \mathbb{R} -linear connection as some trace \mathbb{R} -linear forms on a Lie algebroid.

A *Lie algebroid* is a trip $(A, \rho_A, \llbracket \bullet, \bullet \rrbracket_A)$, in which A is a real vector bundle over a manifold M , $\rho_A : A \rightarrow TM$ (called an *anchor*) is a homomorphism of vector bundles, $(\Gamma(A), \llbracket \bullet, \bullet \rrbracket_A)$ is an \mathbb{R} -Lie algebra and the Leibniz identity

$$\llbracket a, f \cdot b \rrbracket_A = f \cdot \llbracket a, b \rrbracket_A + \rho_A(a)(f) \cdot b \quad \text{for all } a, b \in \Gamma(A), f \in \mathcal{C}^\infty(M)$$

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holds. Since the representation $\varrho : \mathcal{C}^\infty(M) \rightarrow \text{End}_{\mathcal{C}^\infty(M)}(\Gamma(A))$, $\varrho(\nu)(a) = \nu \cdot a$, $\nu \in \mathcal{C}^\infty(M)$, $a \in \Gamma(A)$, is faithful ([9], see also [1]), the anchor induces a homomorphism of Lie algebras $\text{Sec } \rho_A : \Gamma(A) \rightarrow \mathcal{X}(M)$, $a \mapsto \rho_A \circ a$. If ρ_A is a constant rank (i.e. $\text{Im } \rho_A$ is a constant dimensional and completely integrable distribution), we say that $(A, \rho_A, \llbracket \bullet, \bullet \rrbracket_A)$ is *regular*. A tangent bundle TM to a manifold M with the identity as an anchor and the bracket of vector fields is an elementary example of a Lie algebroid. For more about Lie algebroids and their properties we refer for example to [13], [10], [11], [8], [1], [6].

There are Lie functors from many geometric categories to the category of Lie algebroids (see a long list eg in [13], [11]). Especially meaning in the paper have algebroids of vector bundles. We recall that the module $\mathcal{CDO}(E)$ of sections of the Lie algebroid $A(E)$ of a vector bundle E is the space of all covariant differential operators in E , i.e. \mathbb{R} -linear operators $\ell : \Gamma(E) \rightarrow \Gamma(E)$ such that there exists exactly one $\tilde{\ell} \in \mathcal{X}(M)$ with $\ell(f\zeta) = f\ell(\zeta) + \tilde{\ell}(f)\zeta$ for all $f \in \mathcal{C}^\infty(M)$ and $\zeta \in \Gamma(E)$; see for example [14], [13], [11].

Let $(A, \rho_A, \llbracket \bullet, \bullet \rrbracket_A)$ and $(B, \rho_B, \llbracket \bullet, \bullet \rrbracket_B)$ be Lie algebroids over the same manifold M . A homomorphism $\nabla : A \rightarrow B$ of vector bundles is called an *A-connection* in B if $\rho_B \circ \nabla = \rho_A$ (see [1]). If an *A-connection* ∇ in B is a homomorphism of Lie algebroids (∇ preserves the Lie brackets) we say that ∇ is *flat*. The notion of an *A-connection* in B generalizes the known notions of connections (for example usual and partial covariant derivatives in vector bundles, a connection in principal bundles, a connection in extensions of Lie algebroids). In the case where $A = TM$ and $B = A(E)$ is an algebroid of a vector bundle E , *TM-connections* in $A(E)$ are one-to-one with covariant derivatives in E . For an arbitrary Lie algebroid A and $B = A(E)$ we have *A-connections* of E considered in [13], [8], [6]. In case $B = A(P)$ is a Lie algebroid of a principal bundle P , we get *A-connections* in P . In Poisson geometry an especially rule have connections acting from a Lie algebroid T^*M associated to a given Poisson structure. In these examples a connection ∇ considered as a mapping on modules of cross-sections is linear over $\mathcal{C}^\infty(M)$.

By an *\mathbb{R} -linear connection* of A in B we called an \mathbb{R} -linear operator $\nabla : \Gamma(A) \rightarrow \Gamma(B)$ such that

$$\text{Sec } \rho_B \circ \nabla = \text{Sec } \rho_A.$$

An \mathbb{R} -linear connection of A in the Lie algebroid $A(E)$ is called the *\mathbb{R} -linear connection of A on the vector bundle E* . We call the map

$$R^\nabla : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(B), \quad R^\nabla(\alpha, \beta) = \llbracket \nabla_\alpha, \nabla_\beta \rrbracket_B - \nabla_{\llbracket \alpha, \beta \rrbracket_A}$$

a *curvature* of ∇ . We see that $\nabla : \Gamma(A) \rightarrow \Gamma(B)$ is flat if $R^\nabla = 0$. For every Lie algebroid A , the adjoint connection $\text{ad} : \Gamma(A) \rightarrow \mathcal{CDO}(A)$, $\text{ad}(a) = \llbracket a, \bullet \rrbracket_A$ is an \mathbb{R} -linear connection of A on A . The notion of an \mathbb{R} -linear connection includes so-called non-linear connections and connections up to homotopy on super-vector bundles ([5], [6], [7]); such connections have a local property.

Let $(A, \rho_A, \llbracket \bullet, \bullet \rrbracket_A)$, $(B, \rho_B, \llbracket \bullet, \bullet \rrbracket_B)$ be Lie algebroids over a manifold M . An \mathbb{R} -multilinear, antisymmetric map

$$\omega : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_n \longrightarrow \Gamma(B)$$

is called an \mathbb{R} -linear n -form on A with values in B . The space of all such \mathbb{R} -linear n -forms will be denoted by $\mathcal{Alt}_{\mathbb{R}}^n(\Gamma(A); \Gamma(B))$, and the space of \mathbb{R} -linear forms on A with values in B by

$$\mathcal{Alt}_{\mathbb{R}}^{\bullet}(\Gamma(A); \Gamma(B)) = \bigoplus_{k \geq 0} \mathcal{Alt}_{\mathbb{R}}^k(\Gamma(A); \Gamma(B)),$$

where $\mathcal{Alt}_{\mathbb{R}}^0(\Gamma(A); \Gamma(B)) = \Gamma(B)$. Observe that if $\nabla : A \rightarrow B$ is an arbitrary \mathbb{R} -linear connection, then the curvature R^{∇} is an element of $\mathcal{Alt}_{\mathbb{R}}^2(\Gamma(A); \Gamma(B))$. We define the covariant differential operator

$$d_{\mathbb{R}}^{\nabla} : \mathcal{Alt}_{\mathbb{R}}^{\bullet}(\Gamma(A); \Gamma(B)) \longrightarrow \mathcal{Alt}_{\mathbb{R}}^{\bullet+1}(\Gamma(A); \Gamma(B))$$

for \mathbb{R} -linear forms on A with values in B by the classical formula

$$\begin{aligned} (d_{\mathbb{R}}^{\nabla} \eta)(a_1, \dots, a_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \nabla_{a_i} (\eta(a_1, \dots, \hat{a}_i, \dots, a_{n+1})) \\ &\quad + \sum_{i < j} (-1)^{i+j} \eta([a_i, a_j]_A, a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{n+1}). \end{aligned}$$

$d_{\mathbb{R}}^{\nabla}$ is an antiderivation in $\mathcal{Alt}_{\mathbb{R}}^{\bullet}(\Gamma(A); \Gamma(E))$ with respect to the product of \mathbb{R} -linear forms. A flat \mathbb{R} -linear connection $\nabla : \Gamma(A) \rightarrow \Gamma(B)$ induces, denoted by $H_{\nabla, \mathbb{R}}^{\bullet}(A; B)$, the *Lie algebroid \mathbb{R} -cohomology space with coefficients in B* as the cohomology space of the complex $(\mathcal{Alt}_{\mathbb{R}}^{\bullet}(\Gamma(A); \Gamma(B)), d_{\mathbb{R}}^{\nabla})$.

The differential operator $d_{\mathbb{R}}^{\text{Sec } \rho_A}$ induced by the anchor, i.e. by the flat A -connection in TM , will be denoted by $d_{A, \mathbb{R}}$. Since modules $\Gamma(M \times \mathbb{R})$ and $\mathcal{C}^{\infty}(M)$ are isomorphic, it follows that $d_{A, \mathbb{R}}$ is an extension of the exterior derivative from the space $\Omega^{\bullet}(A)$ of $(\mathcal{C}^{\infty}(M)$ -linear) differential forms on A to $\mathcal{Alt}_{\mathbb{R}}^{\bullet}(\Gamma(A); \mathcal{C}^{\infty}(M))$.

Let us recall that the cohomology space of the complex $(\Omega^{\bullet}(A), d_A)$ where $\Omega^{\bullet}(A)$ is the space of all $\mathcal{C}^{\infty}(M)$ -linear forms on A , $d_A = d_{\mathbb{R}}^{\text{Sec } \rho_A} \Big|_{\Omega^{\bullet}(A)} : \Omega^{\bullet}(A) \rightarrow \Omega^{\bullet+1}(A)$, is called the *cohomology of Lie algebroid* and is denoted by $H^{\bullet}(A)$.

Let E be a vector bundle over M . Observe that $\mathcal{Alt}_{\mathbb{R}}^{\bullet}(\Gamma(A); \Gamma(\text{End } E))$ is a left module over the algebra $\mathcal{Alt}_{\mathbb{R}}^{\bullet}(\Gamma(A); \mathcal{C}^{\infty}(M))$ with the standard multiplication of forms. Moreover,

$$\mathcal{Alt}_{\mathbb{R}}^{\bullet}(\Gamma(A); \mathcal{C}^{\infty}(M)) \otimes_{\mathcal{C}^{\infty}(M)} \Gamma(\text{End } E) \cong \mathcal{Alt}_{\mathbb{R}}^{\bullet}(\Gamma(A); \Gamma(\text{End } E))$$

as $\mathcal{C}^{\infty}(M)$ -modules by the isomorphism defined in such a way that

$$\omega \otimes \phi \longmapsto \omega \wedge \phi.$$

In the paper, we define the Chern-Simons forms for \mathbb{R} -linear connections on Lie algebroids. The generalized Chern-Simons formula is derived as a consequence of Stokes' formula for \mathbb{R} -linear forms. The notion of the Chern classes of a vector bundle as cohomology classes of some \mathbb{R} -linear Chern-Simons forms is proposed. We show that such classes for a given \mathbb{R} -linear connection do not depend on the choice of the connection. In the paper, we discuss the wider then in [6] for linear connections set of obstructions to the existence of a flat connection of a given Lie algebroid.

Using ideas from papers Crainic and Fernandes, we introduce the secondary characteristic classes for arbitrary \mathbb{R} -linear connections of Lie algebroids in vector bundles. If an \mathbb{R} -linear A -connection ∇ on a vector bundle E is metrizable with respect to any metric h in E (i.e. $\nabla h = 0$), the defined secondary characteristic classes vanishes. Therefore, secondary characteristic classes of ∇ are obstructions to the existence of an invariant metric with respect to ∇ . In [6] were considered connections up to homotopy (some non-linear connections with a local property). Here we examine all \mathbb{R} -linear connections. At the end of the last section we derive some comments on the Chern-Simons forms for \mathbb{R} -linear connections (in particular for Lie algebroids over odd dimensional manifolds).

2. THE CHERN-SIMONS TRANSGRESSION FORMS ON LIE ALGEBROIDS AND THE CHERN CHARACTER

Let $(A, \rho_A, [\cdot, \cdot]_A)$ be a Lie algebroid on a manifold M , E a vector bundle over M , k a natural number and $\text{pr}_2 : \mathbb{R}^k \times M \rightarrow M$ a projection on the second factor. Consider an \mathbb{R} -linear connection $\nabla : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ of A on E . The standard fibrewise trace $\text{Tr} : \Gamma(\text{End } E) \rightarrow \mathcal{C}^\infty(M)$ on $\text{End}(E)$ induces a trace

$$\text{Tr}_* : \mathcal{Alt}_{\mathbb{R}}^\bullet(\Gamma(A); \Gamma(\text{End } E)) \longrightarrow \mathcal{Alt}_{\mathbb{R}}^\bullet(\Gamma(A); \mathcal{C}^\infty(M))$$

such that $\text{Tr}_*(\omega)(a_1, \dots, a_n) = \text{Tr}((\omega)(a_1, \dots, a_n))$. Set (for $p \geq 1$)

$$\text{ch}_p(\nabla) = \text{Tr}_*(R^\nabla)^p \in \mathcal{Alt}_{\mathbb{R}}^{2p}(\Gamma(A); \mathcal{C}^\infty(M))$$

where $(R^\nabla)^p \in \mathcal{Alt}_{\mathbb{R}}^{2p}(\Gamma(A); \Gamma(\text{End } E))$ is, for $a_1, \dots, a_{2p} \in \Gamma(A)$, given by

$$(R^\nabla)^p(a_1, \dots, a_{2p}) = \frac{1}{2^p} \sum_{\tau \in S_{2p}} \text{sgn } \tau \cdot R_{a_{\tau(1)}, a_{\tau(2)}}^\nabla \circ \dots \circ R_{a_{\tau(2p-1)}, a_{\tau(2p)}}^\nabla.$$

The $2p$ -form $\text{ch}_p(\nabla)$ is called the *Chern character form* associated to ∇ .

Lemma 1. $d_{A, \mathbb{R}} \circ \text{Tr}_* = \text{Tr}_* \circ d_{\mathbb{R}}^{\overline{\nabla}}$ where $\overline{\nabla} : \Gamma(A) \rightarrow \mathcal{CDO}(\text{End } E)$, $\overline{\nabla}_a = [\nabla_a, \bullet]$.

Proof. First, recall that the space $\mathcal{Alt}_{\mathbb{R}}^\bullet(\Gamma(A); \Gamma(\text{End } E))$ is isomorphic to

$$\mathcal{Alt}_{\mathbb{R}}^\bullet(\Gamma(A); \mathcal{C}^\infty(M)) \otimes_{\mathcal{C}^\infty(M)} \Gamma(\text{End } E).$$

Let $\eta \in \mathcal{Alt}_{\mathbb{R}}^n(\Gamma(A); \mathcal{C}^\infty(M))$, $\varphi \in \Gamma(\text{End } E)$. Then $\text{Tr}_*(\eta \otimes \varphi) = \eta \cdot \text{Tr } \varphi$. It is a simple matter to see that $d_{A, \mathbb{R}}(\text{Tr } \varphi) = \text{Tr}_*(d_{\mathbb{R}}^{\overline{\nabla}} \varphi)$. Therefore

$$\begin{aligned} d_{A, \mathbb{R}} \text{Tr}_*(\eta \otimes \varphi) &= d_{A, \mathbb{R}} \eta \cdot \text{Tr } \varphi + (-1)^n \eta \wedge d_{A, \mathbb{R}}(\text{Tr } \varphi) \\ &= d_{A, \mathbb{R}} \eta \cdot \text{Tr } \varphi + (-1)^n \eta \wedge \text{Tr}_*(d_{\mathbb{R}}^{\overline{\nabla}} \varphi) \\ &= \text{Tr}_*(d_{A, \mathbb{R}} \eta \otimes \varphi + (-1)^n \eta \wedge d_{\mathbb{R}}^{\overline{\nabla}} \varphi) \\ &= \text{Tr}_*(d_{\mathbb{R}}^{\overline{\nabla}}(\eta \otimes \varphi)). \end{aligned}$$

□

$\mathcal{C}^\infty(\mathbb{R} \times M)$ -modules $\Gamma(\text{pr}_2^* A)$ and $\mathcal{C}^\infty(\mathbb{R}^k \times M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(A)$ are isomorphic (see [10]) and this way the module of cross-sections of the inverse image

$$\text{pr}_2^*(A) = \{(\gamma, w) \in T(\mathbb{R}^k \times M) \times A : (\text{pr}_2)_* \gamma = \rho_A(w)\} \cong T\mathbb{R}^k \times A$$

of A by pr_2 is a $\mathcal{C}^\infty(\mathbb{R}^k \times M)$ -submodule of

$$\mathcal{X}(\mathbb{R}^k \times M) \times (\mathcal{C}^\infty(\mathbb{R}^k \times M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(A))$$

($T\mathbb{R}^k \times A$ is the Cartesian product of Lie algebroids $T\mathbb{R}^k$ and A , see [12]). We denote cross-sections $0 \times a$, $\frac{\partial}{\partial t^j} \times 0$ of the vector bundle $T\mathbb{R}^k \times A$ briefly by a and $\frac{\partial}{\partial t^j}$, respectively. Let

$$\Delta^k = \left\{ (t_1, \dots, t_k) \in \mathbb{R}^k; \quad \forall i \quad t_i \geq 0, \quad \sum_{i=1}^k t_i \leq 1 \right\}$$

be the *standard k -simplex* in \mathbb{R}^k . Additionally we set the *standard 0-simplex* as $\Delta^0 = \{0\}$. Define

$$\begin{aligned} \int_{\Delta^k} : \mathcal{A}lt_{\mathbb{R}}^\bullet(\Gamma(T\mathbb{R}^k \times A); \mathcal{C}^\infty(\mathbb{R}^k \times M)) &\longrightarrow \mathcal{A}lt_{\mathbb{R}}^{\bullet-k}(\Gamma(A); \mathcal{C}^\infty(M)), \\ \left(\int_{\Delta^k} \omega \right) (a_1, \dots, a_{n-k}) &= \int_{\Delta^k} \omega \left(\frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^k}, a_1, \dots, a_{n-k} \right) \Big|_{(t_1, \dots, t_k, \bullet)} dt_1 \dots dt_k, \\ \left(\int_{\Delta^0} \omega \right) (a_1, \dots, a_n) &= \iota_0^*(\omega(0 \times a_1, \dots, 0 \times a_n)), \quad \int_{\Delta^0} f = \iota_0^* f \end{aligned}$$

for all $n \geq 1$, $1 \leq k \leq n$, $\omega \in \mathcal{A}lt_{\mathbb{R}}^n(\Gamma(T\mathbb{R}^k \times A); \mathcal{C}^\infty(M))$, $f \in \mathcal{C}^\infty(\mathbb{R}^k \times M)$ and where $\iota_0 : M \rightarrow \Delta^0 \times M$ is an inclusion defined by $\iota_0(x) = (0, x)$.

In view of the factorization property in $\mathcal{C}^\infty(\mathbb{R}^k \times M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(A)$, we conclude that for ∇ there exists exactly one \mathbb{R} -linear connection

$$\tilde{\nabla} : \Gamma(T\mathbb{R}^k \times A) \longrightarrow \mathcal{CDO}(\text{pr}_2^* E)$$

of $T\mathbb{R}^k \times A$ on $\text{pr}_2^* E$ such that

$$\left(\tilde{\nabla}_{(X, \sum_i r^i \otimes a^i)} (\nu \circ \text{pr}_2) \right) (t, \bullet) = \nabla_{\sum_i r^i(t, \bullet) \cdot a^i} (\nu)$$

for all $(X, \sum_i r^i \otimes a^i) \in \mathcal{X}(\mathbb{R}^k \times M) \times (\mathcal{C}^\infty(\mathbb{R}^k \times M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(A))$, $\nu \in \Gamma(E)$, $t = (t_1, \dots, t_k) \in \mathbb{R}^k$. In particular, $\left(\tilde{\nabla}_{(0 \times (\rho_A \circ a), 1 \otimes a)} (\nu \circ \text{pr}_2) \right) (t, \bullet) = \nabla_a(\nu)$, $a \in \Gamma(A)$. The connection $\tilde{\nabla}$ is called the *lifting* of ∇ to $T\mathbb{R}^k \times A$.

Let $\nabla^0, \nabla^1, \dots, \nabla^k : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ be \mathbb{R} -linear connections of a Lie algebroid A on a vector bundle E and $\tilde{\nabla}^0, \tilde{\nabla}^1, \dots, \tilde{\nabla}^k : \Gamma(T\mathbb{R}^k \times A) \rightarrow \mathcal{CDO}(\text{pr}_2^* E)$ be their liftings to $T\mathbb{R}^k \times A$. Then there exists an \mathbb{R} -linear connection

$$\nabla^{\text{aff}_k} : \Gamma(T\mathbb{R}^k \times A) \longrightarrow \mathcal{CDO}(\text{pr}_2^* E),$$

called the *affine combination of connections* $\nabla^0, \nabla^1, \dots, \nabla^k$, given by

$$\begin{aligned} &\left(\nabla^{\text{aff}_k}_{(X, \sum_i r^i \otimes a^i)} (\nu \circ \text{pr}_2) \right) (t, \bullet) \\ &= \left(1 - \sum_{i=1}^k t_i \right) \cdot (\nabla^0)_{\sum_i r^i(t, \bullet) \cdot a^i} (\nu) + \sum_{i=1}^k t_i \cdot (\nabla^i)_{\sum_i r^i(t, \bullet) \cdot a^i} (\nu). \end{aligned}$$

For all $0 < k \leq 2p$ we define an \mathbb{R} -linear form

$$\text{cs}_p(\nabla^0, \dots, \nabla^k) = \int_{\Delta^k} \text{ch}_p(\nabla^{\text{aff}_k}) \in \mathcal{A}lt_{\mathbb{R}}^{2p-k}(\Gamma(A); \mathcal{C}^\infty(M))$$

called the *Chern-Simons form* for $(\nabla^0, \dots, \nabla^k)$ and additionally we put $cs_p(\nabla^0) = \text{ch}_p(\nabla^0)$.

We have the following (useful) Stokes' formula for \mathbb{R} -linear forms on A (see [3]) being a generalization of the one for tangent bundles given by R. Bott [4]. For every natural number k ,

$$(2.1) \quad \int_{\Delta^k} \circ d_{T\mathbb{R}^k \times A, \mathbb{R}} + (-1)^{k+1} d_{A, \mathbb{R}} \circ \int_{\Delta^k} = \sum_{j=0}^k (-1)^j \int_{\Delta^{k-1}} \circ (d\sigma_j^{k-1} \times \text{id}_A)^*,$$

where $\sigma_j^k : \mathbb{R}^k \rightarrow \mathbb{R}^{k+1}$ for $0 \leq j \leq k+1$ are functions defined by $\sigma_0^0(0) = 1$, $\sigma_1^0(0) = 0$, and for $t = (t_1, \dots, t_k) \in \mathbb{R}^k$ by

$$\begin{aligned} \sigma_0^k(t) &= \left(1 - \sum_{i=1}^k t_i, t_1, \dots, t_k\right), \\ \sigma_j^k(t) &= (t_1, \dots, t_{j-1}, 0, t_j, \dots, t_k), \quad 1 \leq j \leq k+1, \end{aligned}$$

and where $\left(\left(\int_{\Delta^{k-1}} \circ (d\sigma_j^{k-1} \times \text{id}_A)^*\right) \omega\right)(a_1, \dots, a_{n-k+1})$ is, by definition, equal to

$$\int_{\Delta^{k-1}} \omega \left(d\sigma_j^{k-1} \left(\frac{\partial}{\partial t^1} \right), \dots, d\sigma_j^{k-1} \left(\frac{\partial}{\partial t^{k-1}} \right), a_1, \dots, a_{n-k+1} \right) \Big|_{(t_1, \dots, t_{k-1}, \bullet)} dt_1 \dots dt_{k-1}$$

and

$$\left(\left(\int_{\Delta^0} \circ (d\sigma_j^0 \times \text{id}_A)^* \right) \omega \right) (a_1, \dots, a_n) = (\sigma_j^0 \times \text{id}_M \circ \iota_0)^* (\omega(a_1, \dots, a_n))$$

if $k \geq 2$, $\omega \in \mathcal{Alt}_{\mathbb{R}}^n(\Gamma(T\mathbb{R}^k \times A); \mathcal{C}^\infty(\mathbb{R}^k \times M))$, $a_i \in \Gamma(A)$, $j \in \{0, 1\}$.

The following lemma will be useful below in the proof of the Chern-Simons formula for \mathbb{R} -linear connections of Lie algebroids.

Lemma 2. *Let $a, b \in \Gamma(A)$, $\nu \in \Gamma(E)$, $t \in \mathbb{R}^{k-1}$, $0 \leq j \leq k$, $1 \leq s \leq k$, $1 \leq z \leq k-1$. Denote here the affine combination ∇^{aff_k} of $\nabla^0, \dots, \nabla^k$ by $\nabla^{0, \dots, k}$. Then*

- (a) $\left(R_{a,b}^{\nabla^{0, \dots, k}} (\nu \circ \text{pr}_2) \right) (\sigma_j^{k-1}(t), \bullet) = \left(R_{a,b}^{\nabla^{0, \dots, \hat{j}, \dots, k}} (\nu \circ \text{pr}_2) \right) (t, \bullet),$
- (b) $\left(R_{\frac{\partial}{\partial t^s}, a}^{\nabla^{0, \dots, k}} (\nu \circ \text{pr}_2) \right) (\sigma_j^{k-1}(t), \bullet)$ is equal to $\left(R_{\frac{\partial}{\partial t^s}, a}^{\nabla^{0, \dots, \hat{j}, \dots, k}} (\nu \circ \text{pr}_2) \right) (t, \bullet)$ if $1 \leq s < j$, and $\left(R_{\frac{\partial}{\partial t^{s-1}}, a}^{\nabla^{0, \dots, \hat{j}, \dots, k}} (\nu \circ \text{pr}_2) \right) (t, \bullet)$ if $j \leq s \leq k$, and where \tilde{t}^i are coordinates of the identity map of \mathbb{R}^k ,
- (c) $\left(R_{d\sigma_j^{k-1}(\frac{\partial}{\partial t^z}), a}^{\nabla^{0, \dots, k}} (\nu \circ \text{pr}_2) \right) (\sigma_j^{k-1}(t), \bullet) = \left(R_{\frac{\partial}{\partial t^z}, a}^{\nabla^{1, \dots, k}} (\nu \circ \text{pr}_2) \right) (t, \bullet).$

Proof. Just calculations. □

Theorem 1. (The Chern-Simons formula for Lie algebroids and \mathbb{R} -linear connections) *Let $(A, \rho_A, [\cdot, \cdot])$ be a Lie algebroid on a manifold M , E a vector bundle over M , $k \in \mathbb{N}$, $\nabla^0, \dots, \nabla^k : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ \mathbb{R} -linear connections of A on E . Then*

$$(2.2) \quad (-1)^{k+1} d_{A, \mathbb{R}} (cs_p(\nabla^0, \dots, \nabla^k)) = \sum_{j=0}^k (-1)^j cs_p(\nabla^0, \dots, \widehat{\nabla^j}, \dots, \nabla^k)$$

for all integer numbers p such that $0 < k \leq 2p$ and $d_{A, \mathbb{R}}(cs_p(\nabla^0)) = 0$.

Proof. From Lemma 1 and the Bianchi identity ($d_{\mathbb{R}}^{\overline{\nabla}^j}(R^{\nabla^j}) = 0$) we deduce that forms $\text{ch}_p(\nabla^0)$ and $\text{ch}_p(\nabla^{\text{aff}_k})$ are closed. Since these forms are closed, applying the Stokes formula (2.1) we conclude that

$$(-1)^{k+1} d_{A,\mathbb{R}}(\text{cs}_p(\nabla^0, \dots, \nabla^k)) = \sum_{j=0}^k (-1)^j \int_{\Delta^{k-1}} (d\sigma_j^{k-1} \times \text{id}_A)^* \text{ch}_p(\nabla^{\text{aff}_k}).$$

Let $a_0, \dots, a_{2p-k} \in \Gamma(A)$. From the above

$$\begin{aligned} (-1)^{k+1} d_{A,\mathbb{R}}(\text{cs}_p(\nabla^0, \dots, \nabla^k))(a_0, \dots, a_{2p-k}) \\ = \sum_{j=0}^k (-1)^j \left(\int_{\Delta^{k-1}} (d\sigma_j^{k-1} \times \text{id}_M)^* \text{ch}_p(\nabla^{\text{aff}_k}) \right) (a_0, \dots, a_{2p-k}). \end{aligned}$$

From the definition of $(R^{\nabla^{\text{aff}_k}})^p$ and fact that $R_{\frac{\partial}{\partial \tilde{t}^i}, \frac{\partial}{\partial \tilde{t}^j}}^{\nabla^{\text{aff}_k}} = 0$ (where $(\tilde{t}^1, \dots, \tilde{t}^k)$ is the identity map on the manifold \mathbb{R}^k) we observe that the possible non-zero terms in the above sum are the form

$$R_{d\sigma_j^{k-1}(\frac{\partial}{\partial \tilde{t}^s}), a}^{\nabla^{\text{aff}_k}} \circ \dots \circ R_{b,c}^{\nabla^{\text{aff}_k}} \circ \dots \circ R_{d,e}^{\nabla^{\text{aff}_k}}, \quad a, b, c, d, e \in \Gamma(A).$$

Lemma 2 now yields that $(-1)^{k+1} d_{A,\mathbb{R}}(\text{cs}_p(\nabla^0, \dots, \nabla^k))(a_0, \dots, a_{2p-k})$ is equal to

$$\begin{aligned} \sum_{j=0}^k (-1)^j \int_{\Delta^{k-1}} \text{ch}_p(\nabla^0, \dots, \widehat{\nabla^j}, \dots, \nabla^k) \left(\frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^{k-1}}, a_0, \dots, a_{2p-k} \right) \Big|_{(t_1, \dots, t_{k-1}, \bullet)} dt_1 \dots dt_{k-1} \\ = \left(\sum_{j=0}^k (-1)^j \text{cs}_p(\nabla_0, \dots, \widehat{\nabla^j}, \dots, \nabla_k) \right) (a_0, \dots, a_{2p-k}). \end{aligned}$$

□

Remark 1. If $\nabla^0, \nabla^1, \dots, \nabla^k : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ are $\mathcal{C}^\infty(M)$ -linear connections, then ∇^{aff_k} is a $\mathcal{C}^\infty(M)$ -linear connection. In this case, we obtain a formula due to property of Chern-Simons transgressions in [6] by M. Crainic and R. L. Fernandes.

Corollary 1. Let $\nabla : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ be an \mathbb{R} -linear connection of A on a vector bundle E . The Chern character forms $\text{ch}_p(\nabla) \in \mathcal{Alt}_{\mathbb{R}}^{2p}(\Gamma(A); \mathcal{C}^\infty(M))$ are closed and their cohomology classes

$$\text{ch}_p(A, E) = [\text{ch}_p(\nabla)] \in H_{\rho_A, \mathbb{R}}^{2p}(A; M \times \mathbb{R}),$$

do not depend on the choice of the connection ∇ . Indeed, let $\nabla^0, \nabla^1 : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ be \mathbb{R} -linear connections of A on E . According to (2.2), we have

$$\begin{aligned} d_{A,\mathbb{R}}(\text{cs}_p(\nabla^0, \nabla^1)) &= \text{cs}_p(\nabla^1) - \text{cs}_p(\nabla^0) \\ &= \text{ch}_p(\nabla^1) - \text{ch}_p(\nabla^0). \end{aligned}$$

In this way we have correctly defined the Chern character

$$\text{ch}(A, E) \in H_{\rho_A, \mathbb{R}}(A; M \times \mathbb{R}).$$

Remark 2. ([5], [6]) In the particular case we can obtain the Chern character for a non-linear connection $\nabla : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ of a Lie algebroid A on a vector bundle E , i.e. a local \mathbb{R} -linear connection $\nabla : \Gamma(A) \rightarrow \mathcal{CDO}(E)$. In the space $\Omega_{nl}(A)$ of non-linear differential forms on A (local \mathbb{R} -linear forms on A) we have the differential operator $d_{nl} = d_{A,\mathbb{R}}|_{\Omega_{nl}^\bullet(A)} : \Omega_{nl}^\bullet(A) \rightarrow \Omega_{nl}^{\bullet+1}(A)$.

3. SECONDARY CHARACTERISTIC CLASSES FOR \mathbb{R} -LINEAR CONNECTIONS AND SOME THE CHERN-SIMONS FORMS FOR A PAIR OF CONNECTIONS

Let E be a vector bundle over M with a metric h and $\nabla : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ be an \mathbb{R} -linear connection of a Lie algebroid A on E . We define an \mathbb{R} -linear connection $\nabla^h : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ of A on E such that

$$(\rho_A \circ a)(h(s, t)) = h(\nabla_a s, t) + h(s, \nabla_a^h t), \quad a \in \Gamma(A), \quad s, t \in \Gamma(E).$$

We can observe that

$$R_{a,b}^{\nabla^h} = - (R_{a,b}^\nabla)^*, \quad a, b \in \Gamma(A),$$

where $(R_{a,b}^\nabla)^*$ is the adjoint map to $R_{a,b}^\nabla$ with respect to h . Therefore we obtain the following lemma.

Lemma 3. *If ∇_0, ∇_1 are \mathbb{R} -linear connections of A on E , then*

- (a) $\text{cs}_p(\nabla_0^h) = (-1)^p \text{cs}_p(\nabla_0)$,
- (b) $\text{cs}_p(\nabla_0^h, \nabla_1^h) = (-1)^p \text{cs}_p(\nabla_0, \nabla_1)$.

From the Chern-Simons formula (2.2) and Lemma 3 (a) we deduce that

$$\begin{aligned} d_{A,\mathbb{R}} \text{cs}_p(\nabla, \nabla^h) &= \text{cs}_p(\nabla) - \text{cs}_p(\nabla^h) \\ &= \text{cs}_p(\nabla) - (-1)^p \text{cs}_p(\nabla) \\ &= 0, \end{aligned}$$

because ∇ is flat. In particular, we see that ∇^h is also flat.

Theorem 2. *The cohomology class $[\text{cs}_p(\nabla, \nabla^h)] \in H_{\rho_A, \mathbb{R}}^{2p-1}(A)$ do not depend on the choice of metric h .*

Proof. Let h_1, h_2 be two metrics on E and let ∇^M be any TM -connection on E . Thus $\nabla_o = \nabla^M \circ \rho_A$ is an A -connection on E (i.e. a linear connection). The Chern-Simons formula (2.2) yields

$$(3.1) \quad -d_{A,\mathbb{R}} \text{cs}_p(\nabla, \nabla^{h_j}, \nabla_o^{h_j}) = \text{cs}_p(\nabla^{h_j}, \nabla_o^{h_j}) - \text{cs}_p(\nabla, \nabla_o^{h_j}) + \text{cs}_p(\nabla, \nabla^{h_j})$$

and

$$(3.2) \quad -d_{A,\mathbb{R}} \text{cs}_p(\nabla, \nabla_o, \nabla_o^{h_j}) = \text{cs}_p(\nabla_o, \nabla_o^{h_j}) - \text{cs}_p(\nabla, \nabla_o^{h_j}) + \text{cs}_p(\nabla, \nabla_o)$$

for $j \in \{1, 2\}$. Lemma 3 implies $\text{cs}_p(\nabla^{h_j}, \nabla_o^{h_j}) = (-1)^p \text{cs}_p(\nabla, \nabla_o)$. From this, (3.1) and (3.2) we get

$$\begin{aligned} &\text{cs}_p(\nabla, \nabla^{h_1}) - \text{cs}_p(\nabla, \nabla^{h_2}) \\ &= d_{A,\mathbb{R}} (\text{cs}_p(\nabla, \nabla^{h_2}, \nabla_o^{h_2}) - \text{cs}_p(\nabla, \nabla^{h_1}, \nabla_o^{h_1})) + \text{cs}_p(\nabla, \nabla_o^{h_1}) - \text{cs}_p(\nabla, \nabla_o^{h_2}) \\ &= d_{A,\mathbb{R}} (\text{cs}_p(\nabla, \nabla^{h_2}, \nabla_o^{h_2}) - \text{cs}_p(\nabla, \nabla^{h_1}, \nabla_o^{h_1})) + d_{A,\mathbb{R}} \text{cs}_p(\nabla, \nabla_o, \nabla_o^{h_1}) \\ &\quad - d_{A,\mathbb{R}} \text{cs}_p(\nabla, \nabla_o, \nabla_o^{h_2}) + \text{cs}_p(\nabla_o, \nabla_o^{h_1}) - \text{cs}_p(\nabla_o, \nabla_o^{h_2}). \end{aligned}$$

Because of ∇_o is a linear connection, Proposition 1 from [6] yields $\text{cs}_p(\nabla_o, \nabla_o^{h_1}) - \text{cs}_p(\nabla_o, \nabla_o^{h_2})$ is an exact form. In this way cohomology classes of $\text{cs}_p(\nabla, \nabla^{h_1})$ and $\text{cs}_p(\nabla, \nabla^{h_2})$ are both equal. \square

Definition 1. We call

$$u_{2p-1}(A, E) = [\text{cs}_p(\nabla, \nabla^h)] \in H_{\rho_A, \mathbb{R}}^{2p-1}(A), \quad p \in \{1, \dots, \text{rank } E\},$$

the secondary characteristic classes of an \mathbb{R} -linear connection $\nabla : \Gamma(A) \rightarrow \mathcal{CDO}(E)$.

If there exists in E an invariant metric h with respect to ∇ , then $\nabla^h = \nabla$. Then classes $u_{2p-1}(A, E)$ are equal to zero. Hence these classes are obstructions to the existence of an invariant metric with respect to ∇ .

We obtain the following theorem analogous to Proposition 2 in [6].

Theorem 3. Let ∇, ∇_m be \mathbb{R} -linear connections of A on E and ∇_m be additionally metric.

- (a) If p is even, then $u_{2p-1}(A, E) = 0$.
- (b) If p is odd, then $\text{cs}_p(\nabla, \nabla_m)$ is a closed form and

$$u_{2p-1}(A, E) = [2 \text{cs}_p(\nabla, \nabla_m)].$$

Proof. Let ∇_m be metric connection with respect to a metric h . On account of the Chern-Simons formula (2.2), we have

$$-d_{A, \mathbb{R}} \text{cs}_p(\nabla, \nabla^h, \nabla_m) = \text{cs}_p(\nabla^h, \nabla_m) - \text{cs}_p(\nabla, \nabla_m) + \text{cs}_p(\nabla, \nabla^h).$$

Now Lemma 3 leads to $\text{cs}_p(\nabla^h, \nabla_m) = (-1)^p \text{cs}_p(\nabla, \nabla_m)$, because $\nabla_m^h = \nabla_m$. It follows that

$$\begin{aligned} \text{cs}_p(\nabla, \nabla^h) &= \text{cs}_p(\nabla, \nabla_m) - \text{cs}_p(\nabla^h, \nabla_m) - d_{A, \mathbb{R}} \text{cs}_p(\nabla, \nabla^h, \nabla_m) \\ &= (1 + (-1)^{p+1}) \text{cs}_p(\nabla, \nabla_m) - d_{A, \mathbb{R}} \text{cs}_p(\nabla, \nabla^h, \nabla_m), \end{aligned}$$

which completes the proof. \square

For two \mathbb{R} -linear connections $\nabla^0, \nabla^1 : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ of A on E we define an \mathbb{R} -linear 1-form

$$\lambda = \nabla^1 - \nabla^0 \in \mathcal{Alt}_{\mathbb{R}}^1(\Gamma(A); \Gamma(\text{End } E)).$$

Let us observe that

$$(3.3) \quad R^{\nabla^1} = R^{\nabla^0} + d^{\overline{\nabla^0}} \lambda + [\lambda, \lambda],$$

where $d^{\overline{\nabla^0}}$ is the covariant derivative in $\mathcal{Alt}_{\mathbb{R}}^{\bullet}(\Gamma(A); \Gamma(\text{End } E))$ determined by $\overline{\nabla^0} : \Gamma(A) \rightarrow \mathcal{CDO}(\text{End } E)$, $\overline{\nabla_a^0} = [\nabla_a^0, \bullet]$ for all $a \in \Gamma(A)$, and $[\lambda, \lambda] \in \mathcal{Alt}_{\mathbb{R}}^2(\Gamma(A); \Gamma(\text{End } E))$ is given by $[\lambda, \lambda](a, b) = [\lambda(a), \lambda(b)]$ for all $a, b \in \Gamma(A)$.

Lemma 4. [2] For two \mathbb{R} -linear connections $\nabla^0, \nabla^1 : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ the following properties hold:

$$(3.4) \quad (R^{\nabla^{\text{aff}_1}})_{\frac{\partial}{\partial t}, a}(\nu \circ \text{pr}_2)_{|(t, \bullet)} = \lambda(a)(\nu),$$

$$(3.5) \quad (R^{\nabla^{\text{aff}_1}})_{a, b}(\nu \circ \text{pr}_2)_{|(t, \bullet)} = (1 - t) \cdot R_{a, b}^{\nabla^0}(\nu) + t \cdot R_{a, b}^{\nabla^1}(\nu) + (t^2 - t) \cdot [\lambda, \lambda]_{(a, b)}(\nu)$$

for all $a, b \in \Gamma(A)$, $\nu \in \Gamma(E)$, $t \in \mathbb{R}$.

The Chern-Simons forms of the first and the second rank.

Let $\theta \in \mathcal{Alt}_{\mathbb{R}}^1(\Gamma(A); \Gamma(\text{End } E))$, $\nabla : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ be an \mathbb{R} -linear connection of A on E . Therefore $\nabla + \theta$ is also an \mathbb{R} -linear A -connection on E , and $\text{cs}_1(\nabla, \nabla + \theta) \in \mathcal{Alt}_{\mathbb{R}}^1(\Gamma(A); \mathcal{C}^\infty(M))$ is given by $\text{cs}_1(\nabla, \nabla + \theta)(a) = \text{tr}(\theta(a))$, $a \in \Gamma(A)$. Moreover, we conclude from (3.4), (3.5) and (3.3) that

$$\text{tr} \left(R^{\nabla^{\text{aff}_1}} \right)^2 \left(\frac{\partial}{\partial t}, \bullet \right)_{|(t, \bullet)} = 2 \text{tr} \left(\theta \wedge R^{\nabla^0} + t \cdot \theta \wedge d_{\mathbb{R}}^{\overline{\nabla^0}} \theta + t^2 \cdot \theta \wedge \theta \wedge \theta \right)$$

for all $a_1, a_2, a_3 \in \Gamma(A)$, $t \in \mathbb{R}$, hence

$$\text{cs}_2(\nabla, \nabla + \theta) = \text{tr} \left(2\theta \wedge R^{\nabla} + \theta \wedge d_{\mathbb{R}}^{\overline{\nabla}} \theta + \frac{2}{3} \theta \wedge \theta \wedge \theta \right).$$

If ∇ and $\nabla + \theta$ are both flat, then $d_{\mathbb{R}}^{\overline{\nabla}} \theta = -\theta \wedge \theta$, which then yields

$$\text{cs}_2(\nabla, \nabla + \theta) = -\frac{1}{3} \text{tr}(\theta \wedge \theta \wedge \theta).$$

For every manifold M of an odd dimension $2m - 1$, $\text{cs}_m(\nabla, \nabla + \theta)$ is closed. In the case where M is a 3-dimensional manifold, $\text{cs}_2(\nabla, \nabla + \theta)$ is closed and is given by the above formula; if additionally ∇ is flat, we see that

$$(3.6) \quad \text{cs}_2(\nabla, \nabla + \theta) = \text{tr} \left(\theta \wedge d_{\mathbb{R}}^{\overline{\nabla}} \theta + \frac{2}{3} \theta \wedge \theta \wedge \theta \right).$$

(3.6) is a generalization of the known formula for tangent bundles of smooth, compact, oriented, three dimensional manifolds and standard connections (see for example [15]) to arbitrary rank three vector bundles and \mathbb{R} -linear connections.

Moreover, we add (see [2]) that if both \mathbb{R} -linear connections $\nabla^0, \nabla^1 : \Gamma(A) \rightarrow \mathcal{CDO}(E)$ of a Lie algebroid A on a vector bundle E are flat, then the Chern-Simons \mathbb{R} -linear form $\text{cs}_p(\nabla^0, \nabla^1)$ is equal to $(-1)^{p+1} \frac{p!(p-1)!}{(2p-1)!} \text{Tr}_*(\lambda^{2p-1})$. In particular, for any flat \mathbb{R} -linear connection ∇ of A on E , ∇^h is also flat and we conclude that the class $u_{2p-1}(A, E)$ is represented by the form

$$(-1)^{p+1} \frac{p!(p-1)!}{(2p-1)!} \text{Tr}_*(\omega^{2p-1}),$$

where $\omega = \nabla^h - \nabla \in \mathcal{Alt}_{\mathbb{R}}^1(\Gamma(A); \Gamma(\text{End } E))$.

REFERENCES

- [1] B. BALCERZAK, J. KUBARSKI AND W. WALAS, *Primary characteristic homomorphism of pairs of Lie algebroids and Mackenzie algebroid*, Banach Center Publ. **54** (2001), 135–173.
- [2] B. BALCERZAK, *Modular classes of Lie algebroids homomorphisms as some the Chern-Simons forms*, Univ. Iagel. Acta Math. **47** (2009), 11–28.
- [3] B. BALCERZAK, *The Generalized Stokes theorem for \mathbb{R} -linear forms on Lie algebroids*, Accepted to Journal of Applied Analysis; available as preprint arXiv:1102.2594, 2011.
- [4] R. BOTT, *Lectures on characteristic classes and foliations*, Springer Lecture Notes in Math. 279, Springer, Berlin, 1972.
- [5] M. CRAINIC, *Connections up to homotopy and characteristic classes*, Preprint arXiv, arXiv:math/0010085v2 (2000).

- [6] M. CRAINIC AND R. L. FERNANDES, *Secondary Characteristic Classes of Lie Algebroids*, In: Quantum Field Theory and Noncommutative Geometry, Lecture Notes in Phys. 662, pp. 157–176, Springer, Berlin, 2005.
- [7] S. EVENS, J. H. LU AND A. WEINSTEIN, *Transverse measures, the modular class and a cohomology pairing for Lie algebroids*, Q. J. Math. **50** (1999), 417–436.
- [8] R. L. FERNANDES, *Lie algebroids, holonomy and characteristic classes*, Adv. in Math. **170** (2002), 119–179.
- [9] J.-C. HERZ, Pseudo-algèbres de Lie, *C. R. Math. Acad. Sci. Paris* **263** (1953), I, 1935–1937, and II, 2289–2291.
- [10] PH. J. HIGGINS AND K. C. H. MACKENZIE, *Algebraic constructions in the category of Lie algebroids*, J. Algebra **129** (1990), 194–230.
- [11] J. KUBARSKI, *The Chern-Weil homomorphism of regular Lie algebroids*, Publ. Dép. Math., Nouv. Sér., Univ. Claude Bernard, Lyon, 1991, 1–69.
- [12] J. KUBARSKI, Invariant cohomology of regular Lie algebroids, in: *Analysis and Geometry in Foliated Manifolds* (Proceedings of the VII International Colloquium on Differential Geometry, Santiago de Compostella, Spain, 26–30 July 1994), pp. 137–151, World Sci. Publ., Singapore–New Jersey–London–Hong Kong, 1995.
- [13] K. C. H. MACKENZIE, *General Theory of Lie Groupoids and Lie Algebroids*, London Math. Soc. Lecture Note Ser. 213, Cambridge Univ. Press, 2005.
- [14] N. TELEMAN, *A characteristic ring of a Lie algebra extension*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (8) Mat. Appl., vol. **52** (1972), 498–506 and 708–711.
- [15] W. ZHANG, *Lectures on Chern-Weil Theory and Witten Deformations*, Nankai Tracts Math., vol. 4, World Sci. Publ., New Jersey–London–Singapore–Hong Kong, 2001.

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